Photon trajectories in the Kerr-Newman metric

1980 J. Phys. A: Math. Gen. 13 3213

(http://iopscience.iop.org/0305-4470/13/10/018)

J. Phys. A: Math. Gen. 13 (1980) 3213–3219. Printed in Great Britain

Photon trajectories in the Kerr–Newman metric

M Calvani†, F de Felice‡ and L Nobili‡

† Institute of Astronomy, University of Padova, Italy
‡ Institute of Physics, University of Padova, Italy

Received 13 February 1980

Abstract. We give a more detailed description of the null trajectories in the Kerr–Newman metric. Interesting results which were not expected from what was known in the uncharged case are (a) a considerable enhancement of the energy storage in a restricted region around the source in the naked singularity case and (b) the singularity is completely disconnected from the asymptotic region in the sense that no photon can reach it even in the equatorial plane.

1. Introduction

The motion of test particles in the Kerr gravitational field has been extensively investigated in the last few years, and the results are of crucial importance in understanding what happens around black holes (Sharp 1979).

The Kerr–Newman metric, which describes the gravitational and electromagnetic field of a charged and rotating collapsed object, did not stimulate the same interest because it is quite probable that black holes have no substantial charge, although this is far from certain (Ruffini and Treves 1973, Harrison 1976).

Because of that we consider further investigation of the properties of this metric appropriate; our purpose here is to give a complete description of the photon trajectories in the equatorial plane of the K–N metric and to study the existence and stability of the spherical null orbits.

2. The equations of motion

The first-order equations of motion for a charged particle in the Kerr–Newman metric were derived by Carter (1968); the equations describing the radial and angular motion are

$$\Sigma^2\dot{r}^2 = R = [E(r^2+a^2) - al - eQr]^2 - \Delta(m^2r^2+K) \tag{1}$$

$$\Sigma^2\dot{\theta}^2 = \Theta = K - m^2a^2\cos^2\theta - (1/\sin^2\theta)[Ea\sin^2\theta - l]^2 \tag{2}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2 + Q^2 = (r^2+a^2) + T$$

$$T = Q^2 - 2Mr$$

and we define

$$A = (r^2 + a^2)^2 - a^2\Delta.$$

The quantities M, Q and a describe respectively the mass, charge and specific angular momentum of the source; m and e are the mass and charge of the test particle. The quantities E, l and K are constants of the motion; E describes the particle total energy at infinity, l is the axial component of the particle angular momentum and K is related to the square of its total angular momentum (de Felice 1980). The latter is always non-negative and is connected to the Carter constant $\mathcal{Q}$ by

$$K \equiv \mathcal{Q} + (l - aE)^2$$

(Misner *et al* 1973 p 899).

The equations of motion for the photons are obtained from (1) and (2) by putting

$$m = e = 0. \tag{3}$$

The K–N geometry describes a black hole only if $M^2 \geqslant a^2 + Q^2$; in this case the horizons are at

$$r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}$$

where $\Delta = 0$. When $M^2 < a^2 + Q^2$ the metric describes instead a naked singularity (at $r = 0$, $\theta = \pi/2$) and Δ is always positive.

3. Equatorial null trajectories

The equatorial trajectories of photons are conveniently investigated searching for the locus of the turning points, which are solutions of the equation $R = 0$. From equations (1) and (3) and the condition for equatorial motion $K = (l - aE)^2$ or $\mathcal{Q} = 0$ we have

$$(\Delta - a^2)l^2 - 2Tal - A = 0. \tag{4}$$

The constants of motion are designed in units of the photon energy: $l \to l/E$ and $K \to K/E^2$. We solve equation (4) with respect to l and study the function $l = l(r, a, Q, M)$; we have

$$l_\pm = (aT \pm r^2\Delta^{1/2})/(\Delta - a^2). \tag{5}$$

Equation (5) represents a three-parameter (a, Q, M) family of curves in the l–r plane. To know how the curves $l_\pm$ change varying the parameters, let us first study the function

$$Q^2 = r(2M - r) \equiv Q_1^2 \tag{6}$$

which is the solution of $(\Delta - a^2) = 0$. For a chosen value of Q it gives the divergences of $l_\pm$ and, at the same time, the intersections of the ergosphere with the equatorial plane. Note that the ergosphere, which is defined by $(r^2 + Q^2 + a^2\cos^2\theta - 2Mr) = 0$, exists only when $Q^2 < M^2$.

The function

$$Q^2 = r(2M - r) - a^2 \equiv Q_2^2 = Q_1^2 - a^2 \tag{7}$$

is the solution of $\Delta = 0$; when $M^2 \leqslant (a^2 + Q^2)$ it defines the horizons and also the locus where $l_\pm = l_-$.

The function

$$Q^2 = 2Mr \equiv Q_3^2 \tag{8}$$

is the solution of $T = 0$, i.e. $r = Q^2/2M$. This surface is interesting because there the K–N metric acquires a flat space–time form without being flat; it is relevant in the studies of the repulsive phenomena which a particle experiences in the vicinity of a naked singularity (Cohen and Gautreau 1979, de Felice *et al* 1980).

The function

$$Q^2 = r[r(r^2 + a^2) + 2Ma^2]/a^2 \equiv Q_4^2 \tag{9}$$

is the solution of $A = 0$, that is where one of $l_\pm$ goes to zero. Moreover it is the place where $g_{\phi\phi} = 0$ in the equatorial plane; recall that $g_{\phi\phi} < 0$ is a necessary condition for causality violation.

Figure 1 shows the functions $Q_1^2 - Q_4^2$, assuming that $M^2 > a^2$; similar graphs can be easily drawn for $M^2 < a^2$. For a chosen value of the parameters of the source it is now straightforward to draw the functions $l_\pm(r)$, that is the locus of the inversion points. This is done in figure 2, taking into account also the properties

$$\lim_{r\to\infty} l_\pm = \pm\infty \qquad l_\pm(r = 0) = a.$$

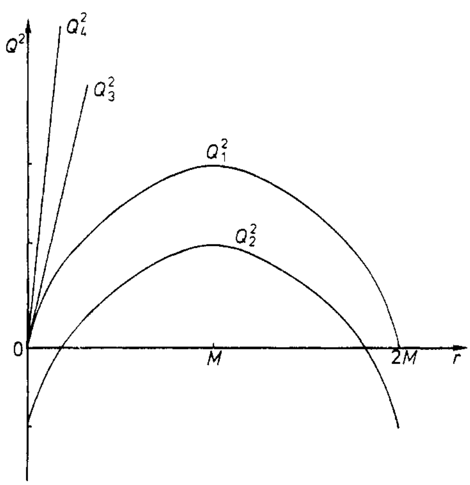


Figure 1. The curves Q_1^2, Q_2^2, Q_3^2 and Q_4^2 are shown, not to scale, assuming $M > a$.

Figures 2(a) and 2(b) refer to naked singularities ($M^2 < a^2 + Q^2$), while figure 2(c) refers to black holes. In particular figure 2(a) holds when there is no ergosphere ($Q^2 > M^2$); figure 2(b) holds when there is an ergosphere but no horizons ($M^2 > Q^2 > (M^2 - a^2)$), and figure 2(c) when there are both an ergosphere and horizons.

These graphs may be compared and contrasted with similar ones for the Kerr metric (de Felice 1968, Helliwell and Mallinckrodt 1975, Calvani and de Felice 1978) and the Tomimatsu–Sato metrics (Tomimatsu and Sato 1973, Calvani and Catenacci 1976).

The main features of the motion of the photons (in the equatorial plane) which are due to the charge Q are the following.

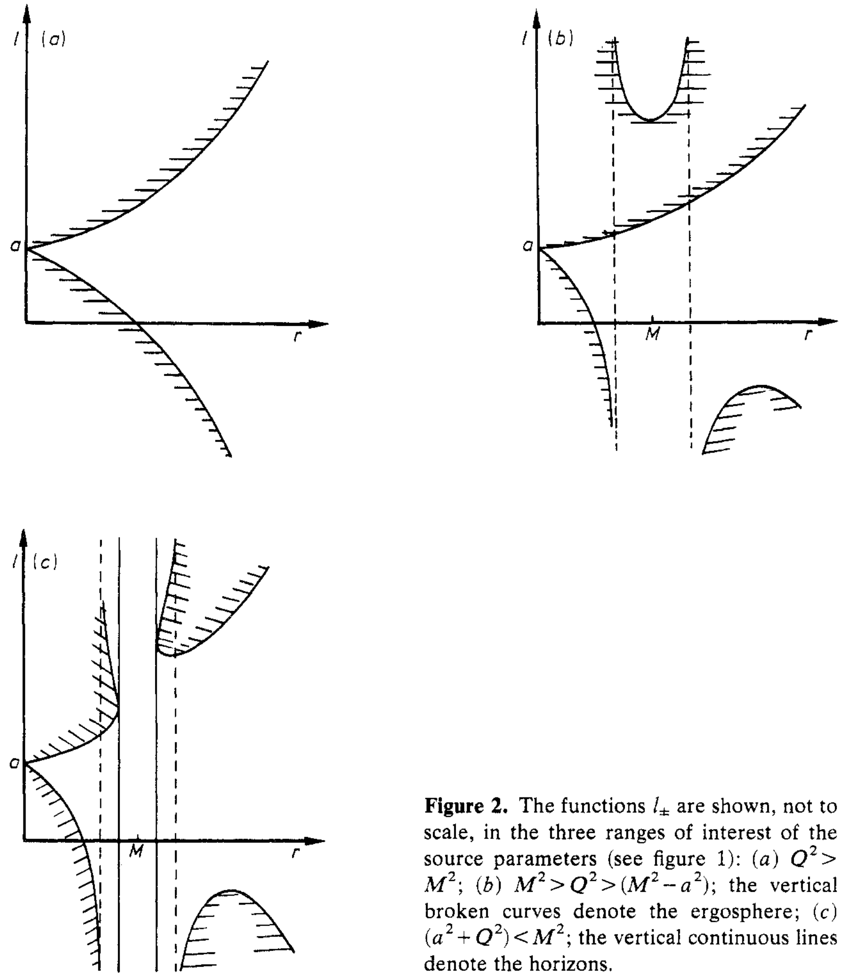


Figure 2. The functions $l_{\pm}$ are shown, not to scale, in the three ranges of interest of the source parameters (see figure 1): (*a*) $Q^2 > M^2$; (*b*) $M^2 > Q^2 > (M^2 - a^2)$; the vertical broken curves denote the ergosphere; (*c*) $(a^2 + Q^2) < M^2$; the vertical continuous lines denote the horizons.

(i) The existence of bound orbits for naked singularities when $M^2 > Q^2 > (M^2 - a^2)$, see figure 2(*b*); a similar situation arises in the T–S metrics when $a > M$ and is connected with the surfaces of infinite redshift.

(ii) While in the Kerr metric the singularity can be reached by photons coming in from infinity only when $l \geqslant a$, in the K–N metric no photon succeeds in reaching $r = 0$, except for those with $l = a$.

(iii) Photons with $l = a$ deserve a particular attention, and they will be considered in more detail in the following section.

4. Spherical null orbits

The existence of time-like spherical orbits (that is orbits of constant radius) in the field of a black hole was shown by Wilkins (1972). These orbits cross the equatorial plane repeatedly and in the limit of large radius go asymptotically to Keplerian circles, while near the horizon they have a helix-like shape.

Similar orbits, but of the null type, exist in the field of the Kerr naked singularity (Calvani and de Felice 1978): the singularity is surrounded by shells of photons which, remarkably, gather around the surface $r = M$; this surface is the position of the horizon in the extreme black hole case $a = M$.

Let us now extend the same analysis to the K–N metric. The radial coordinate of a null orbit will be constant at some value $r = r_0$ if the following conditions are simultaneously satisfied:

$$R(r_0) = 0 \qquad \left.\frac{\partial R}{\partial r}\right|_{r=r_0} = 0 \tag{10}$$

or, from equations (1) and (3):

$$(r^2 + a^2 - al)^2 - \Delta K = 0 \tag{11}$$

$$-2alr + 2r(r^2 + a^2) - (r - M)K = 0. \tag{12}$$

The solutions of (11) and (12) are

$$K = \frac{4\Delta r^2}{(r-M)^2} \equiv K_- \qquad l = \frac{1}{a}\left[r^2 + a^2 - \frac{2\Delta r}{(r-M)}\right] \equiv l_- \tag{13}$$

$$K = 0 \equiv K_+ \qquad l = (r^2 + a^2)/a \equiv l_+. \tag{14}$$

Let us study first K_-; the extrema of the curve $K_-(r, a, M, Q)$ in the K–r plane are along

$$Q^2 = (r/M)(r^2 - 3Mr + 3M^2) - a^2 \equiv Q_5^2, \tag{15}$$

and Q_5^2 is zero along

$$a^2 = (r/M)(r^2 - 3Mr + 3M^2) \equiv a_2^2 \tag{16}$$

which is shown in figure (3). Figures (4*a*) and (*b*) show Q_5^2, Q_2^2 and Q_1^2, respectively for $a < M$ and $a > M$. Note that at $r = M$, $Q_5^2 = Q_2^2 = (M^2 - a^2)$.

With the aid of figures 3 and 4 it is now easy to draw the curve K_- (figure 5); one can prove that the full part of the curve K_- denotes stable orbits ($\partial^2 R/\partial r^2 < 0$), while the dashed part denotes unstable orbits ($\partial^2 R/\partial r^2 > 0$). It is remarkable that, as for the Kerr naked singularity (Calvani and de Felice 1978), the surface $r = M$ is surrounded by stable orbits.

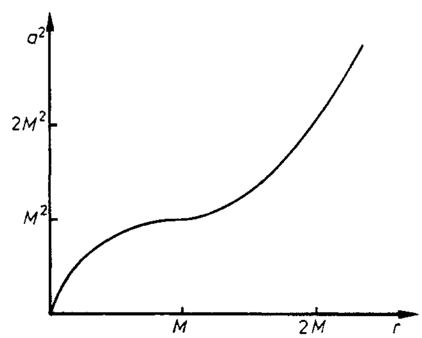


Figure 3. The function a_2^2 is shown; for a chosen value of a it gives the zeros of Q_5^2.

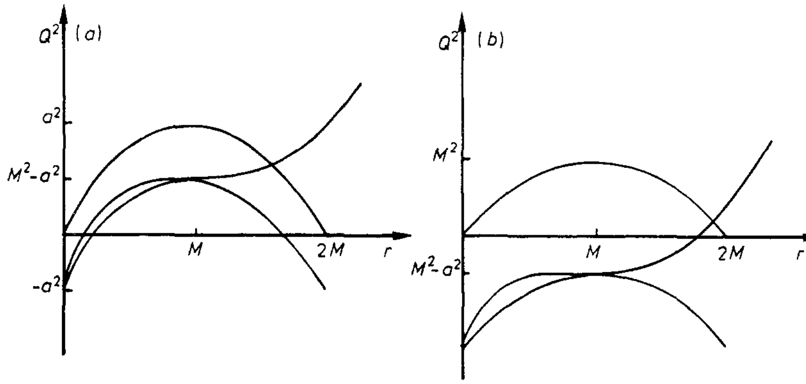
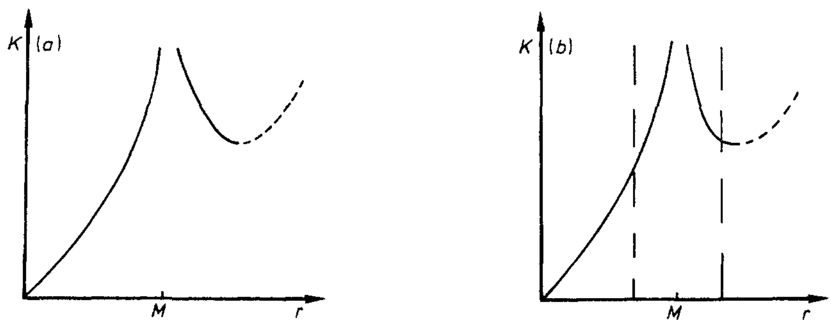


Figure 4. The curves Q_1^2, Q_2^2 and Q_5^2 are shown for (a) $M > a$; (b) $M < a$. The zeros of Q_5^2 are deduced from figure 3; Q_5^2 gives the extrema of K_-.



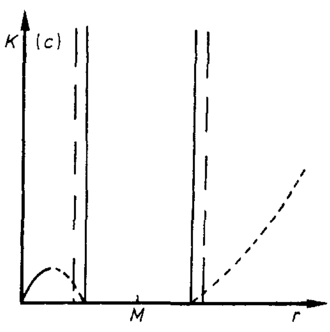


Figure 5. The locus of spherical photon trajectories (i.e. the function K_-) is shown in the $K-r$ plane, not to scale, for (a) $Q^2 > M^2$; (b) $M^2 > Q^2 > (M^2 - a^2)$; $(c)(a^2 + Q^2) < M^2$. The extrema are deduced from Q_5^2 in figure 4.

The case $K_+ = 0$ deserves a particular attention. From equation (2) it follows that real photon trajectories with $K = 0$ can only have $\dot{\theta} = 0$, and therefore must move on hyperboloids $\theta = \text{constant}$ with $\sin^2 \theta = l/a$, which implies $l \leq a$ (vortical motion) (de Felice and Calvani 1972, Bicak and Stuchlik 1976). From equation (1) it follows that $R = 0$ only for $r = 0$, $\theta = \pi/2$ and $l = a$, that is on the ring singularity itself where the equations of motion lose their meaning. We believe that these photons can only exist confined in the singularity itself (Calvani *et al* 1978); in fact these photons play a key role in some models of the Kerr metric source (Hamity 1976, Israel 1977, Bernstein 1978).

5. Conclusions

The most interesting effect which is due to the charge in the K–N metric is undoubtedly that of confining some photons in a bounded region around the source even in the equatorial plane. This is a phenomenon of energy storage which was known to exist in the Kerr metric (Calvani and de Felice 1978) as well as in some internal solutions (de Felice 1969, Kuchowicz 1974, Guha Thakurta 1978).

A further consequence of the charge is the impossibility for the K–N singularity to be reached by photons coming in from infinity in the equatorial plane. This means that, even in the naked singularity conditions, the singularity ($r = 0$, $\theta = \pi/2$) is completely disconnected from the asymptotic regions, contrary to what happens in the Kerr metric; in the latter case the singularity could in fact be reached from infinity by a photon moving only in the equatorial plane. This property suggests that the results that a naked singularity of the Kerr type would be indistinguishable from a black hole to far distant observations (Calvani *et al* 1978) is even more true in the Kerr–Newman case.

Acknowledgments

Work supported by the Italian National Council of Research (CNR). Thanks are due to Dr N Sharp for useful comments.

References

Bernstein M 1978 *PhD Thesis*, University of Indiana
Bicak J and Stuchlik Z 1976 *Bull. Astr. Inst. Czech.* **27** 129
Calvani M and Catenacci R 1976 *Nuovo Cim.* **31** B 41
Calvani M and de Felice F 1978 *Gen. Rel. Grav.* **9** 889
Calvani M, de Felice F and Nobili L 1978 *Lett. Nuovo Cim.* **23** 539
Carter B 1968 *Phys Rev.* **174** 1559
Cohen J M and Gautreau R 1979 *Phys. Rev.* D **19** 2273
de Felice F 1968 *Nuovo Cim.* **57** B 351
—— 1969 *Nuovo Cim.* **63** B 649
—— 1980 *J. Phys. A: Math. Gen.* **13** 1701
de Felice F and Calvani M 1972 *Nuovo Cim.* **10** B 447
de Felice F, Nobili L and Calvani M 1980 *J. Phys. A: Math. Gen.* **13** to be published.
Guha Thakurta S N 1978 *J. Phys. A: Math. Gen.* **11** 2213
Hamity V 1976 *Phys. Lett.* **56** A 77
Harrison E R 1976 *Nature* **264** 525
Helliwell T M and Mallinckrodt A J 1975 *Phys. Rev.* D **12** 2993
Israel W 1977 *Phys. Rev.* D **15** 935
Kuchowicz B 1974 *Gen. Rel. Grav.* **5** 201
Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)
Ruffini R and Treves A 1973 *Astrophys. Lett.* **13** 109
Sharp N 1979 *Gen. Rel. Grav.* **10** 659
Tomimatsu A and Sato H 1973 *Prog. Theor. Phys.* **50** 95
Wilkins D C 1972 *Phys. Rev.* D **5** 814